Troisième Cycle de Physique en Suisse Romande
Exercises for Lecture 1
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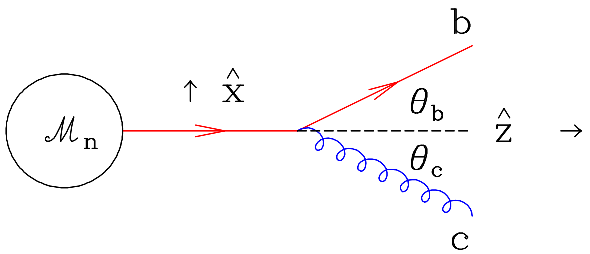
1) Show that if both (massless) quarks have positive helicity the associated current for small θ is

$$J^\mu = \bar{u}_b \gamma^\mu u_a = \sqrt{E_a E_b}(2, \theta_b, i\theta_b, 2) \tag{1}$$

with θ_b as shown in the figure. Hence show that for small opening angle θ, the



matrix element squared for gluon polarization in the plane is

$$|\mathcal{M}_{\rm in}|^2 \propto \left|\frac{J_\mu \varepsilon^\mu_{\rm in}}{t}\right|^2 = \frac{(1+z)^2}{(1-z)}\,\frac{1}{t} \tag{2}$$

whilst for polarization out of the plane

$$|\mathcal{M}_{\rm out}|^2 \propto \left|\frac{J_\mu \varepsilon^\mu_{\rm in}}{t}\right|^2 = (1-z)\,\frac{1}{t} \tag{3}$$

so that for

$$\frac{1}{2}\left[|\mathcal{M}_{\rm in}|^2 + |\mathcal{M}_{\rm out}|^2\right] \propto \frac{(1+z^2)}{(1-z)}\,\frac{1}{t} \tag{4}$$

2) The plus prescription is defined as

$$\int_0^1 dz\; f(z)\; g(z)_+ = \int_0^1 dz\; (f(z) - f(1))\; g(z) \tag{5}$$

Show that

$$\int_0^1 dz\; z^{(n-1)} \frac{1}{(1-z)_+} = -\sum_{j=1}^{n-1} \frac{1}{j} \tag{6}$$

Using this result show that the moments of the splitting functions

$$\begin{aligned}
P_{qq}(x) &= C_F\left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right], \\
P_{qg}(x) &= T_R\left[x^2 + (1-x)^2\right], \quad T_R = \frac{1}{2},
\end{aligned}$$

$$
\begin{aligned}
P_{gq}(x) &= C_F\left[\frac{1+(1-x)^2}{x}\right] , \\
P_{gg}(x) &= 2C_A\left[\frac{x}{(1-x)_+}+\frac{1-x}{x}+x\,(1-x)\right] \\
&\quad + \delta(1-x)\frac{(11C_A-4n_fT_R)}{6} , \qquad (7)
\end{aligned}
$$

are

$$
\begin{aligned}
\gamma_{qq}^{(0)}(n) &= C_F\left[-\frac{1}{2}+\frac{1}{n(n+1)}-2\sum_{k=2}^{n}\frac{1}{k}\right] , \\
\gamma_{qg}^{(0)}(n) &= T_R\left[\frac{(2+n+n^2)}{n(n+1)(n+2)}\right] , \\
\gamma_{gq}^{(0)}(n) &= C_F\left[\frac{(2+n+n^2)}{n(n^2-1)}\right] , \\
\gamma_{gg}^{(0)}(n) &= 2C_A\left[-\frac{1}{12}+\frac{1}{n(n-1)}+\frac{1}{(n+1)(n+2)}-\sum_{k=2}^{n}\frac{1}{k}\right]-\frac{2}{3}n_fT_R . \qquad (8)
\end{aligned}
$$

Hence show that for $n=2$ the DGLAP takes the form

$$
t\frac{\partial}{dt}\begin{pmatrix}\Sigma(2,t)\\ g(2,t)\end{pmatrix}=\frac{\alpha_S(t)}{2\pi}\begin{pmatrix}-\frac{4}{3}C_F & \frac{1}{3}n_f\\ \frac{4}{3}C_F & -\frac{1}{3}n_f\end{pmatrix}\begin{pmatrix}\Sigma(2,t)\\ g(2,t)\end{pmatrix} . \qquad (9)
$$

The eigenvectors and associated eigenvalues of this system of equations are

$$
\begin{aligned}
O^+(2,t) &= \Sigma(2,t)+g(2,t) \quad \text{with eigenvalue } 0 , \\
O^-(2,t) &= \Sigma(2,t)-\frac{n_f}{4C_F}g(2,t) \quad \text{with eigenvalue } -\left(\frac{4}{3}C_F+\frac{n_f}{3}\right) . \qquad (10)
\end{aligned}
$$

where

$$
\Sigma(x,t)=\sum_i[q_i(x,t)+\bar{q}_i(x,t)] \qquad (11)
$$

and $\Sigma(n,t), q(n.t)$ and $g(n,t)$ are the corresponding moments.

$$
f(n,t)=\int_0^1 dz\; z^{n-1}f(z) \qquad (12)
$$